Mathematics 1 AESB1110-15: Test 2 - ANSWERS

October 21, 2016

Rules:

- No points are assigned for a question if only the final answer is given without any intermediate steps.
- Subtract 0.25 p. for the first occurrence of an arithmetic error. Do not subtract further points if the same error 'propagates' into subsequent calculations.

Answer to Question 1:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \quad \boxed{+1\,p.}$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + o(x^{13}) \quad \boxed{+1\,p.}$$

Answer to Question 2:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-ax} = 1 + (-ax) + \frac{(-ax)^2}{2!} + \frac{(-ax)^3}{3!} + \dots = 1 - ax + \frac{a^2}{2!}x^2 + \frac{(e^{-ax})^{(3)}_{x=c}}{3!}x^3$$

$$= 1 - ax + \frac{a^2}{2!}x^2 - a^3\frac{e^{-ac}}{3!}x^3 \quad \boxed{+0.5\,p.}$$

Introducing: $c = \theta x,\ 0 \leq \theta \leq 1$

$$\text{error} = |r_3(x)| = \left|e^{-ax} - \left(1 - ax + \frac{a^2}{2!}x^2\right)\right| = \left|-a^3\frac{e^{-a\theta x}}{3!}x^3\right| = \left|e^{-a\theta x}\right| \left|\frac{a^3}{3!}x^3\right|$$

$$\left|e^{-a\theta x}\right| \leq 1, \quad \text{since} \quad 0 \leq \theta \leq 1,\ a > 0,\ x \geq 0$$

Therefore, we obtain $\quad \text{error} = |r_3(x)| \leq \dfrac{a^3}{3!}|x|^3 \quad \boxed{+0.5\,p.}$

We want: error < 0.001

Hence, formally, we would choose the interval of x, so that, $\frac{a^3}{3!}|x|^3 < 0.001$,

$$|x|^3 < \frac{3!}{a^3} 0.001 \quad \Rightarrow \quad |x| < \left(\frac{3!}{a^3} 0.001\right)^{1/3} \quad \boxed{+0.5\,p.}$$

However, since $x \geq 0$,

the correct range of x is: $0 \leq x < \left(\frac{3!}{a^3} 0.001\right)^{1/3}$ $\boxed{+0.5\,p.}$

Answer to Question 3:

$$\begin{aligned} f'(x) &= \left[\int_{x^2}^{\cos^2(x)} \ln(t^2)\,dt\right]' = \left[\int_{x^2}^{a} \ln(t^2)\,dt + \int_{a}^{\cos^2(x)} \ln(t^2)\,dt\right]' \\ &= \left[-\int_{a}^{x^2} \ln(t^2)\,dt + \int_{a}^{\cos^2(x)} \ln(t^2)\,dt\right]' \quad \boxed{+0.5\,p.} \\ &= -\ln\left((x^2)^2\right)(x^2)' + \ln\left((\cos^2(x))^2\right)\left(\cos^2(x)\right)' \\ &= -2x\ln(x^4) - 2\cos(x)\sin(x)\ln(\cos^4(x)) \quad \boxed{+1.5\,p.} \\ &= -8x\ln(|x|) - 8\cos(x)\sin(x)\ln(|\cos(x)|) \end{aligned}$$

Answer to Question 4:

$$\int \sin^{-1}(x)\,dx = \begin{bmatrix} u = \sin^{-1}(x), & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}}dx, & v = x \end{bmatrix} = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx \quad \boxed{+1\,p.}$$

$$\begin{aligned} &= \begin{bmatrix} u = 1 - x^2, & du = -2x\,dx \\ x\,dx = -\frac{1}{2}\,du & \end{bmatrix} = x\sin^{-1}(x) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= x\sin^{-1}(x) + \frac{1}{2}\frac{1}{(1/2)}u^{1/2} + C = x\sin^{-1}(x) + \sqrt{1-x^2} + C \quad \boxed{+1\,p.} \end{aligned}$$

Answer to Question 5: The integrand is discontinuous at $x = 1$.

$$\begin{aligned} &\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \lim_{t\to1+} \int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx \quad \boxed{+0.5\,p.} \\ &= \lim_{t\to1+} \left[\frac{3}{2}(x-1)^{2/3}\right]_t^9 = \lim_{t\to1+} \left[\frac{3}{2}(9-1)^{2/3} - \frac{3}{2}(t-1)^{2/3}\right] \quad \boxed{+1\,p.} \\ &= \frac{3}{2}8^{2/3} - 0 = 6 \quad \boxed{+0.5\,p.} \end{aligned}$$